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ANALYTIC METHOD FOR CALCULATING AERODYNAMIC FORCES IN A THREE-DIMENSIONAL PROBLEM IN CONDITIONS OF THE "LOCALIZABILITY LAW"

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An analytic solution of the problem of determination of aerodynamic forces (drag, lift and side force) acting on an arbitrary three-dimensional body whose motion satisfies the localization law is derived. It is assumed in the localization law that the body is of convex shape and that the momentum acting on a surface element depends only on conditions of flow and the local angle between the velocity and the normal to the surface. This law is successfully applied in many domains of aerodynamics and dynamics of flight. Particular cases of the localization law are, for instance, various modifications of Newtonian law of air resitance, laws which determine the action of rarefied gas on a body flying at supersonic velocity, and the effect of light pressure on a body. As an example, the problem of determination of aerodynamic properties of an elliptic cone is considered.

1. Let us consider the flow around a body in conditions of the localization law [1], i.e. we consider the body to be of convex shape and that the momentum acting on a surface element depends only on the conditions of flow and on the local angle between the inner normal $\mathbf{n}$ to the surface and the unit vector of the stream velocity $\mathbf{v}$.

According to this theory the dimensionless momentum normalized with respect to the dynamic head $\rho V^2 / 2$ or its projections on the natural coordinates: local pressure p and shearing stress τ can be represented in the form

$$p = \sum_{k=1}^{R} A_k (\mathbf{v}\cdot\mathbf{n})^k, \quad \tau = (\mathbf{v}\cdot\mathbf{t}) \sum_{k=1}^{R-1} B_k (\mathbf{v}\cdot\mathbf{n})^k \tag{1.1}$$

where $\mathbf{t}$ is the unit vector of the tangent to the surface element, lying in the plane of vectors $\mathbf{v}$ and $\mathbf{n}$, R is the order of the approximating polynomial, and $(A_k, B_k) = f_k(M, Re, T_w)$ are coefficients which depend on conditions of flow around the body and can be obtained with the use of known theoretical or experimental methods.

We select the system of independent angles α and φ (such system was used in [1] for the case of flow around bodies of revolution) which is defined by the relationships

$$\cos\alpha = (\mathbf{v}\cdot\mathbf{i}_1), \quad \operatorname{tg}\varphi = -(\mathbf{v}\cdot\mathbf{i}_3)/(\mathbf{v}\cdot\mathbf{i}_2)$$

where $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are the unit vectors of the system of axes attached to the body. Then

$$\mathbf{v} = \mathbf{i}_1\cos\alpha + \mathbf{i}_2\sin\alpha\cos\varphi - \mathbf{i}_3\sin\alpha\sin\varphi \tag{1.2}$$

The dimensionless coefficient of the total aerodynamic force acting on a three-dimensional body moving in a flow of gas or light is

$$\mathbf{c}_F = c_x\mathbf{v} + c_y\mathbf{j} + c_z\mathbf{k} \tag{1.3}$$

where c_x, c_y and c_z are the dimensionless coefficients of drag, lift and side force, respectively, and $\mathbf{v}$, $\mathbf{j} = \mathbf{v}_\alpha / |\mathbf{v}_\alpha|$ and $\mathbf{k} = \pm \mathbf{v}_\varphi / |\mathbf{v}_\varphi|$ are unit vectors of the velocity coordinate system.

Differential relationships were derived for c_F in [1]. For the determination of coefficients of aerodynamic forces for an arbitrary three-dimensional body, whose motion satisfies the assumptions of the localizability law and the arbitrary expansions (1.1) of the local momentum taken from [1], we select variables α and φ. Then, using formulas (1.2) and (1.3) and introducing the "reduced" drag coefficient

$$c_x^1(\alpha, \varphi) = c_x(\alpha, \varphi) + N_R \tag{1.4}$$

we obtain the second order equation in partial derivatives of the elliptic kind

$$L[c_x^1] = \csc^2 \alpha c^1_{x_{\varphi\varphi}} + c^1_{x_{\alpha\alpha}} + \operatorname{ctg} \alpha c^1_{x_\alpha} + (R+1)(R+2) c_x^1 = (R+1)\Psi_R \tag{1.5}$$

and the relationship

$$c^1_{x_\alpha} = (R+1) c_y, \qquad c^1_{x_\varphi} = -(R+1) c_z \sin\alpha \tag{1.6}$$

where

$$c^1_{x_\delta} = \partial c_x^1 / \partial\delta \quad (\delta = \alpha, \varphi), \qquad \Psi_R = \Phi_R + (R+2) N_R$$

$$N_R = \frac{1}{S_R} \int_{S^*} (\mathbf{v}\cdot\mathbf{n}) \Big\{ \frac{R-1}{2} A_1 (\mathbf{v}\cdot\mathbf{n}) + \sum_{k=1}^{R-1} \Big[(A_{k+1} - B_k) \frac{R-k-1}{k+2} (\mathbf{v}\cdot\mathbf{n})^{k+1} - B_k (\mathbf{v}\cdot\mathbf{n})^{k-1} \Big] \Big\} dS$$

$$\Phi_R = \frac{1}{S_R} \int_{S^*} \Big\{ A_1 + (R-1) A_1 (\mathbf{v}\cdot\mathbf{n})^2 + \sum_{k=1}^{R-1} [(A_{k+1} - B_k)\{(R-k-1)(\mathbf{v}\cdot\mathbf{n})^{k+2} + (k+1)(\mathbf{v}\cdot\mathbf{n})^k\} + (R+2) B_k (\mathbf{v}\cdot\mathbf{n})^k] \Big\} dS$$

S_R is a characteristic area, and integration is carried out over the "illuminated" area of surface S^* which is determined by the condition $(\mathbf{vn}) \geqslant 0$.

Note that the substitution (1.4) considerably improves the smoothness of the right-hand part of the equation for c_x, which is important for solving this equation. The derived system of equations makes it possible to include aerodynamic forces in the general system of dynamic equations. These formulas can be used in experimental and theoretical investigations, since with one of the components of aerodynamic forces determined, it is easy to calculate the remaining components by formulas (1.6).

The system (1.5), (1.6) makes it possible to derive the solution for the problem of determination of aerodynamic forces acting on three-dimensional bodies in the entire range of angles α and φ. This solution can also be used in problems of optimization and establishment of generalized similarity laws under conditions of the localizability law [2].

2. The determination of aerodynamic properties of an arbitrary three-dimensional body reduces to the solution of Eq. (1.5). In mathematical terms this problem can be considered as one of finding function $c_x^1(\alpha, \varphi)$ which is continuous and bounded at all points of a sphere of unit radius.

In variables α and φ which are analogs of a spherical system of coordinates the sphere is transformed into a rectangle with boundaries $0 \leqslant \varphi \leqslant 2\pi$ and $0 \leqslant \alpha \leqslant \pi$.

The conditions of periodicity of solution at coincident meridians $\varphi = 0$ and $\varphi = 2\pi$ of the sphere and the conditions of solution boundedness at the poles of the sphere $\alpha = 0$ and $\alpha = \pi$ of the form

$$c_x^1(\alpha, \varphi)\big|_{\varphi=0} = c_x^1(\alpha, \varphi)\big|_{\varphi=2\pi}, \qquad c^1_{x_\varphi}\big|_{\varphi=0} = c^1_{x_\varphi}\big|_{\varphi=2\pi} \tag{2.1}$$
$$\big|\lim_{\alpha\to 0} c_x^1(\alpha, \varphi)\big| < \infty, \qquad \big|\lim_{\alpha\to\pi} c_x^1(\alpha, \varphi)\big| < \infty$$

can be taken as appropriate boundary conditions (of the problem).

Solution of the boundary value problem (1.5), (2.1) will be derived by the method of expansion in eigenfunctions of the related homogeneous problem.

Equations for eigenvalues and eigenfunctions of operator $L[c_x^1]$ are of the form

$$c^1_{x_{\varphi\varphi}} \csc^2\alpha + c^1_{x_{\alpha\alpha}} + c^1_{x_\alpha} \operatorname{ctg}\alpha + (R+1)(R+2)\,c_x^1 + \lambda c_x^1 = 0 \tag{2.2}$$

The boundary value problem (2.1), (2.2) is a particular case of the boundary value problem for the elliptic equation, and the boundary conditions (2.1) of this problem are equivalent to homogeneous boundary conditions.

Equation (2.2) admits the separation of variables, hence we seek a solution of the form $c_x^1(\alpha, \varphi) = w(\varphi)v(\alpha)$. Separating variables in Eq. (2.2) and boundary conditions (2.1), we obtain for function $w(\varphi)$ the Sturm-Liouville problem with the condition of solution periodicity at the ends of the integration interval $\varphi\,[0, 2\pi]$

$$w''(\varphi) + \mu w(\varphi) = 0$$
$$w(\varphi)\big|_{\varphi=0} = w(\varphi)\big|_{\varphi=2\pi}, \qquad w'(\varphi)\big|_{\varphi=0} = w'(\varphi)\big|_{\varphi=2\pi}$$

The eigenvalues of this problem $\mu_0 = 0$, $\mu_m = m^2$ $(m = 1, 2, \ldots)$ and the corresponding to these eigenfunctions

$$w_0(\varphi) = 1, \qquad w_m(\varphi) = \begin{cases} \sin m\varphi \\ \cos m\varphi \end{cases} \quad (m = 1, 2, \ldots)$$

constitute the complete system of orthogonal functions along segment $\varphi\,[0, 2\pi]$ and are bounded along the latter.

For each μ_m function $v(\alpha)$ represents a particular case of the homogeneous boundary value problem in eigenvalues with the condition of boundedness at the ends of the integration interval $\alpha[0, \pi]$

$$v''(\alpha) + \operatorname{ctg}\alpha \cdot v'(\alpha) + \left[(R+1)(R+2) + \lambda - \frac{m^2}{\sin^2\alpha}\right] v(\alpha) = 0 \tag{2.3}$$
$$\big|\lim_{\alpha\to 0} v(\alpha)\big| < \infty, \qquad \big|\lim_{\alpha\to\pi} v(\alpha)\big| < \infty \tag{2.4}$$

The substitution $x = \cos\alpha$ reduces Eq. (2.3) to the general Legendre equation whose solution exists for any complex λ and m [3]. For positive integral m solutions which satisfy boundary conditions (2.4) exist only for

$$\lambda_n = n(n+1) - (R+1)(R+2), \qquad n = m, m+1, m+2, \ldots \tag{2.5}$$

and are "adjoint" Legendre polynomials of the m-th order

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}, \quad n = m, m+1, m+2, \ldots$$

($P_n(x)$ denote ordinary Legendre polynomials). Thus the boundary value problem (2.3), (2.4) has for each m the eigenvalues λ_n (2.5) and the related system of orthogonal eigenfunctions. Hence the complete set of eigenvalues of the linear operator $L[c_x^1]$ for boundary conditions is defined by

$$\lambda_n = n(n+1) - (R+1)(R+2), \qquad n = 0, 1, 2, \ldots \tag{2.6}$$

To that set corresponds the complete closed system of eigenfunctions $c^1_{x_{m,n}}$

$$c^1_{x_{0,n}} = P_n(\cos\alpha), \quad c^1_{x_{m,n}} = P_n^m(\cos\alpha)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix}, \quad n = 0, 1, 2, \ldots; \tag{2.7}$$
$$m = 1, 2, \ldots, n$$

which are orthogonal in region $0 \leqslant \alpha \leqslant \pi$ and $0 \leqslant \varphi \leqslant 2\pi$.

3. To solve the nonhomogeneous equation (1.5) with boundary conditions we expand function $\Psi_R(\alpha,\varphi)$ in region $0 \leqslant \varphi \leqslant 2\pi$ and $0 \leqslant \alpha \leqslant \pi$ into a convergent Laplace series in eigenfunctions (2.7), using the completeness of eigenfunctions in that region and the reasonable smoothness of function $\Psi_R(\alpha, \varphi)$. We obtain

$$\Psi_R(\alpha,\varphi) = \sum_{n=0}^{\infty} b_{0,n} P_n(\cos\alpha) + \sum_{n=1}^{\infty}\sum_{m=1}^{n}(b_{m,n}\cos m\varphi + a_{m,n}\sin m\varphi)\, P_n^m(\cos\alpha) \tag{3.1}$$

$$b_{0,n} = \frac{2n+1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \Psi_R(\alpha,\varphi)\, P_n(\cos\alpha)\sin\alpha\, d\varphi\, d\alpha$$

$$a_{m,n} = \frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_0^{2\pi} \Psi_R(\alpha,\varphi)\, P_n^m(\cos\alpha)\sin m\varphi \sin\alpha\, d\varphi\, d\alpha \tag{3.2}$$

$$b_{m,n} = \frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_0^{2\pi} \Psi_R(\alpha,\varphi)\, P_n^m(\cos\alpha)\cos m\varphi \sin\alpha\, d\varphi\, d\alpha$$

$$m = 1, 2, \ldots, n$$

Solution of the nonhomogeneous problem is also sought in the form of a Laplace series with undetermined coefficients $c_{m,n}$ and $d_{m,n}$

$$c_x^1(\alpha,\varphi) = \sum_{n=0}^{\infty} c_{0,n} P_n(\cos\alpha) + \sum_{n=1}^{\infty}\sum_{m=1}^{n}(c_{m,n}\cos m\varphi + d_{m,n}\sin m\varphi)\, P_n^m(\cos\alpha) \tag{3.3}$$

Since the eigenvalue $\lambda_{R+1} = 0$, hence, as shown in [4], the problem has no solution for an arbitrary $\Psi_R(\alpha,\varphi)$ in the right-hand part of Eq. (1.5). For a solution to exist it is necessary to impose on function $\Psi_R(\alpha,\varphi)$ the supplementary condition for the coefficients in the expansion of that function into series (3.2), corresponding to λ_{R+1}, to vanish, i.e.

$$b_{0,R+1} = b_{m,R+1} = a_{m,R+1} = 0, \quad m = 1, 2, \ldots, R+1 \tag{3.4}$$

The analysis of function $\Psi_R(\alpha,\varphi)$ presented above and specific calculations show that for the considered class of problems function Ψ_R satisfies condition (3.4). Hence any arbitrary linear combination of eigenfunctions (2.7) which correspond to the eigenvalue $\lambda_{R+1} = 0$ and is representable in the form

$$Y_{R+1} = k_{0,R+1} P_{R+1}(\cos\alpha) + \sum_{m=1}^{R+1} (k_{m,R+1} \cos m\varphi + H_{m,R+1} \sin m\varphi)\, P_{R+1}^m(\cos\alpha) \tag{3.5}$$

where $k_{0,R+1}$, $k_{m,R+1}$ and $H_{m,R+1}$ are arbitrary constants, will satisfy the considered problem.

It is obvious that function Y_{R+1} is in essence a general solution of the related homogeneous problem (1.5), (2.1). The solution of the boundary value problem (1.5), (2.1) becomes nonunique and generally contains $[2(R+1)+1]$ constants of integration which have to be determined by the physical conditions of a specific problem.

The remaining coefficients of solution (3.3) are determined by the coefficients (3.2) of expansion of function $\Psi_R(\alpha,\varphi)$ and eigenvalues λ_n using formulas

$$c_{m,n} = \frac{(R+1)\, b_{m,n}}{[(R+1)(R+2) - n(n+1)]}, \quad d_{m,n} = \frac{(R+1)\, a_{m,n}}{[(R+1)(R+2) - n(n+1)]} \tag{3.6}$$

Consequently, the final solution of the problem of determination of the coefficient of drag for an arbitrary three-dimensional body, which satisfies the localization law for an arbitrary form of expansion of local momentum (1.1) throughout the variation range of angles α and φ $(0 \leqslant \varphi \leqslant 2\pi$ and $0 \leqslant \alpha \leqslant \pi)$, is of the form

$$c_x^1(\alpha,\varphi) = Y_{R+1} + \sum_{\substack{n=0 \\ n\neq R+1}}^{\infty} c_{0,n} P_n(\cos\alpha) + \sum_{\substack{n=1 \\ n\neq R+1}}^{\infty} \sum_{m=1}^{n} (c_{m,n}\cos m\varphi + d_{m,n}\sin m\varphi)\, P_n^m(\cos\alpha) \tag{3.7}$$

Note that a solution of similar form but with another system of variables was presented in [5] for a particular case (Newton's law of air resistance) without the general statement and analysis of the boundary value problem, when expansion (1.1) contains only one nonzero coefficient $A_2 \neq 0$. Constants of integration in formula (3.7) can be determined by solving the system of linear algebraic equations which is derived from (3.7) for known values of its left-hand part obtained from theoretical or experimental data at $[2(R+1)+1]$ points.

The analysis of calculations carried out for specific bodies shows that the determination of constants of integration by using experimental data compensates to a considerable extent the inaccuracy of the "localization method". Additional conditions related to the flow symmetry considerably simplify solution (3.7) and reduce the number of constants of integration.

4. The majority of real aerodynamic bodies contain one or more planes of symmetry. If $\varphi = 0$ and $\varphi = \pi$ define the plane of symmetry of the considered body, the nonsymmetric terms in solution (3.7) and formulas (3.1) and (3.5) vanish $a_{m,n} = d_{m,n} =$

$H_{m,n} = 0$, the solution contains $(R+2)$ constants of integration, and is of the form

$$c_x^1(\alpha, \varphi) = \sum_{m=0}^{R+1} k_{m,R+1} \cos m\varphi P_{R+1}^m (\cos\alpha) + \sum_{\substack{n=0 \\ n\neq R+1}}^{\infty} \sum_{m=0}^{n} c_{m,n} \cos m\varphi P_n^m (\cos\alpha) \quad (4.1)$$

If the plane $\varphi = \pm\pi/2$ is that of the body symmetry, the solution is of the form

$$c_x^1(\alpha, \varphi) = \sum_{m=0}^{[(R+1)/2]} k_{2m,R+1} \cos 2m\varphi P_{R+1}^{2m} (\cos\alpha) + \quad (4.2)$$
$$\sum_{m=0}^{[R/2]} H_{2m+1,R+1} \sin(2m+1)\varphi P_{R+1}^{2m+1} (\cos\alpha) +$$
$$\sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{[n/2]} c_{2m,n} \cos 2m\varphi P_n^{2m} (\cos\alpha) + \right.$$
$$\left. \sum_{m=0}^{[(n-1)/2]} d_{2m+1,n} \sin(2m+1)\varphi P_n^{2m+1} (\cos\alpha) \right\} \quad n \neq R+1$$

The number of coefficients in solution (4.2) required for calculations reduces to half of that needed in the general solution (3.7). The number of constants of integration $(k_{2m,R+1}, H_{2m+1,R+1})$ is equal $(R+2)$. If the considered body has two planes of symmetry $\varphi = 0, \pi$ and $\varphi = \pm\pi/2$, solution (3.7) is considerably simplified (the second and fourth sums in (4.2) disappear) and the number of constants of integration is reduced to $[1 + R/2]$.

It should be pointed out that the described general method of derivation of solution (3.7) for the nonhomogeneous elliptic equation (1.5) with boundary conditions (2.1) does not exclude the possibility of obtaining a different form of solution of that equation, when function $\Psi_R(\alpha,\varphi)$ in the right-hand side of the equation is of a special form which admits a direct determination of the particular solution $F(\alpha,\varphi)$ of Eq. (1.5) which satisfies conditions (2.1). In that case the solution of the boundary value problem (1.5), (2.1) can be presented in the form

$$c_x^1(\alpha,\varphi) = Y_{R+1} + F(\alpha, \varphi)$$

5. As an example of the determination of aerodynamic properties of a three-dimensional body we present the results of calculations for an elliptic cone (Fig. 1).

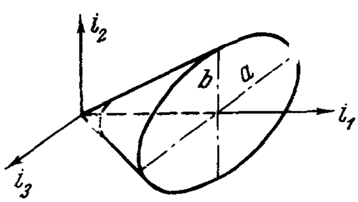

Fig. 1

To simplify computations the case of $R = 2$ and $A_0 = A_1 = 0$ was considered. Expansion (1.1) of local momentum is of the form

$$p = A_2 (\mathbf{v}\cdot\mathbf{n})^2, \qquad \tau = B_1 (\mathbf{v}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{t})$$

Coefficients $A_2 = 2(2-\sigma)$ and $B_1 = 2\sigma_\tau$ (where σ and σ_τ are the accommodation coefficients of normal and tangent momenta, respectively) correspond to conditions of free-molecule flow of rarefied gas with diffusion-mirror reflection pattern. Coefficients $A_2 = k$ and $B_1 = 0$ correspond to the Newtonian hypersonic flow.

In the range of angles α and φ in which the stream flows over the whole surface of the elliptic cone the solution for the "reduced" drag coefficient $c_x^1(\alpha, \varphi)$ is of the form

$$c_x^1(\alpha, \varphi) = k_0 P_3(\cos\alpha) + k_2 \cos 2\varphi P_3^2(\cos\alpha) + {}^3/_5 (A_2 - B_1)\cos\alpha$$

$$P_3(\cos\alpha) = {}^1/_2 (5\cos^3\alpha - 3\cos\alpha), \quad P_3^2(\cos\alpha) = 15\cos\alpha\sin^2\alpha$$

Since $N_2 = -B_1\cos\alpha$, hence, in accordance with (1.4), the solution for the physical drag coefficient $c_x(\alpha, \varphi)$ is of the form

$$c_x(\alpha, \varphi) = k_0 P_3(\cos\alpha) + k_2 \cos 2\varphi P_3^2(\cos\alpha) + {}^1/_5 (3A_2 + 2B_1)\cos\alpha$$

The constants of integration k_0 and k_2 are determined by known theoretical or experimental values $c_x^\circ(\alpha = 0)$ and $c_x^*(\alpha = \alpha^*, \varphi = 0)$ and $P_3(\cos\alpha^*) = 0$. We have

$$k_0 = c_x^\circ - \frac{1}{5}(3A_2 + 2B_1), \quad k_2 = \frac{c_x^* - {}^1/_5 (3A_2 + 2B_1)\cos\alpha^*}{15\sin^2\alpha^*\cos\alpha^*}$$

Formulas (1.6) yield for the coefficients of lift c_y and side force c_z the expressions

$$c_y(\alpha, \varphi) = \frac{1}{3} c^1_{x_\alpha} = \sin\alpha \left\{ \frac{k_0}{2}(1 - 5\cos^2\alpha) + 5k_2\cos 2\varphi(\cos^2\alpha + \cos 2\alpha) - \frac{1}{5}(A_2 - B_1) \right\}$$

$$c_z(\alpha, \varphi) = -\frac{1}{3\sin\alpha} c^1_{x_\varphi} = 10k_2\sin 2\varphi\cos\alpha\sin\alpha$$

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